MATH 10550, EXAM 3 SOLUTIONS

1. In finding an approximate solution to the equation $x^3+2x-4=0$ using Newton's method with initial approximation $x_1 = 1$, what is x_2?
Solution. Recall that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Hence,

$$x_2 = 1 - \frac{(1)^3 + 2(1) - 4}{3(1)^2 + 2(1)} = 1 - \frac{-1}{5} = \frac{6}{5}$$

2. A box with a square base and open top must have a volume of 32 cm^3. Find the dimensions of the box that minimizes the surface area of the box.
Solution. Let the edge length of the base of the box be s, and the

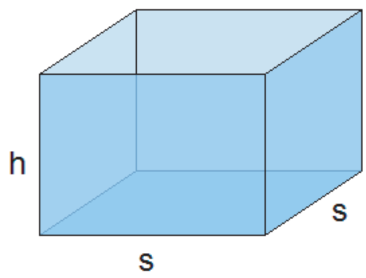

FIGURE 1

height of the box be h. The volume of the box is given by $V = hs^2$ and the surface area is given by $A = s^2 + 4sh$. Since we require $32 = V = hs^2$, we get that $h = \frac{32}{s^2}$, and thus $A = s^2 + 128s^{-1}$. To find the minimum of $A(s)$ - the surface area, we need to find the critical numbers of the function $A(s)$. Solving the equation

$$0 = A' \Rightarrow 0 = 2s - \frac{128}{s^2} \Rightarrow 128 = 2s^3 \Rightarrow 64 = s^3 \Rightarrow s = 4,$$

gives the critical point $s = 4$. Another critical point occurs at $s = 0$, but we are only interested in s on the interval $(0, \infty)$. Furthermore, $A'(1) = -126 < 0$ and $A'(8) = 14 > 0$, and so by the First Derivative Test, $s = 4$ is a local minimum. Since $A(s)$ is continuous and differentiable on $(0,\infty)$ and $s = 4$ is the only local extrema, it must be a global minimum. Thus the box will have minimal surface area when $s = 4$. Hence, the dimensions of the box that minimize its surface area are $4 \times 4 \times 2$.

3. Calculate the following indefinite integral

$$\int \frac{x^2 - 2\sqrt{x}}{x} dx =$$

Solution.

$$\begin{aligned}
\int \frac{x^2 - 2\sqrt{x}}{x}\, dx &= \int (x - 2x^{-1/2})\, dx \\
&= \int x\, dx - 2\int x^{-1/2}\, dx \\
&= \frac{1}{2}x^2 - 4\sqrt{x} + C.
\end{aligned}$$

4. Calculate the following definite integral

$$\int_{-\pi/2}^{\pi/2} |\sin x| dx =$$

Solution. Recall that $\sin x < 0$ when $-\frac{\pi}{2} \leq x < 0$ and $\sin x \geq 0$ when $0 \leq x \leq \frac{\pi}{2}$. Thus,

$$|\sin x| = \begin{cases} -\sin x & \text{if} \quad -\frac{\pi}{2} \leq x < 0 \\ \sin x & \text{if} \quad 0 \leq x \leq \frac{\pi}{2} \end{cases}$$

Therefore, we have that

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} |\sin x| dx &= \int_{-\pi/2}^{0} (-\sin x)\, dx + \int_{0}^{\pi/2} \sin x\, dx \\
&= \Big[\cos x\Big|_{-\pi/2}^{0} + \Big[-\cos x\Big|_{0}^{\pi/2} \\
&= \left(\cos(0) - \cos\left(\frac{-\pi}{2}\right)\right) + \left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right) \\
&= (1-0) + (0+1) = 2.
\end{aligned}$$

5. What is the indefinite integral

$$\int \frac{\cos x}{(\sin x - 1)^2}\, dx =?$$

Solution. Use substitution. Let u=$\sin x - 1$; then $du = \cos x\ dx$. Thus we have that

$$\int \frac{\cos x}{(\sin x - 1)^2}\,dx = \int \frac{1}{u^2}du = \int u^{-2}du = -u^{-1}+C = -(\sin x - 1)^{-1}+C.$$

6. The equation of the slant asymptote of the curve $y = \dfrac{2x^3 + x^2 + 3}{x^2 + 1}$ is:

Solution. Using long division to divide x^2+1 into $2x^3+x^2+3$, we get $\frac{2x^3+x^2+3}{x^2+1} = 2x+1-\frac{2x+2}{x^2+1}$. We then have that $\frac{2x^3+x^2+3}{x^2+1} - 2x+1 = \frac{2x+2}{x^2+1}$. Since $\lim_{x\to\infty} \frac{2x+2}{x^2+1} = 0$, we know that the slant asymptote is $y = 2x+1$.

7. A table of values for a function f is given below.

t	0	2	4	6
$f(t)$	1	2	4	5

Use 3 rectangles and right endpoints to estimate the value of the integral

$$\int_0^6 f(t)\,dt.$$

Solution. Each of the 3 rectangles will have width $\frac{6-0}{3} = 2$. Since we are using right endpoints, we have

$$\int_0^6 f(t)\,dt \approx 2(f(2)+f(4)+f(6)) = 2(2+4+5) = 22.$$

8. Let $g(x) = \displaystyle\int_0^{(\sin x)^3} \sqrt{1+t^2}\,dt$. Find $g'(x)$.

Solution. Letting $u = (\sin x)^3$ we have,

$$g(x) = \int_0^{(\sin x)^3} \sqrt{1+t^2}dt = \int_0^u \sqrt{1+t^2}dt.$$

Thus, by the Chain Rule and The Fundamental Theorem of Calculus Part I

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx} = \frac{d\left(\int_0^u \sqrt{1+t^2}dt\right)}{du}\frac{d(\sin x)^3}{dx}$$

$$= \sqrt{1+u^2}\cdot 3(\sin x)^2\cos x = \sqrt{1+(\sin x)^6}\cdot 3(\sin x)^2\cos x.$$

9. Calculate the integral $\displaystyle\int_0^2 3x^2\sqrt{x^3+1}\,dx$

Solution. Let $u = x^3 + 1$, and thus $du = 3x^2 dx$. Then at $x = 0$, we have $u = 0^3 + 1 = 1$, an dat $x = 2$, we have $u = 2^3 + 1 = 9$. Making these substitutions, we have

$$\begin{aligned} \int_0^2 3x^2\sqrt{x^3+1}\,dx &= \int_1^9 \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_1^9 = \frac{2}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) \\ &= \frac{2}{3}\left(3^3 - 1\right) = \frac{2}{3}(26) = \frac{52}{3}. \end{aligned}$$

10. Find a Riemann sum corresponding to the integral $\int_0^1 \cos x dx$
Solution. $\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$. Thus, the Reimann sum is

$$\sum_{i=1}^n f(x_i)\Delta x = \sum_{i=1}^n \cos\left(\frac{i}{n}\right) \cdot \frac{1}{n} = \frac{1}{n}\sum_{i=1}^n \cos\left(\frac{i}{n}\right).$$

11. (a) Evaluate the definite integral $\int_0^2 x^3 dx$ using the **definition** of the definite integral.

Hint: $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Solution. We have $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, and using right-hand endpoints, $x_i = a + i\Delta x = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}$. Thus

$$\begin{aligned} \int_0^2 x^3 dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n \left(\frac{2i}{n}\right)^3 \cdot \frac{2}{n} = \lim_{n\to\infty}\sum_{i=1}^n \frac{8i^3}{n^3}\cdot\frac{2}{n} \\ &= \lim_{n\to\infty}\frac{16}{n^4}\sum_{i=1}^n i^3 = \lim_{n\to\infty}\frac{2^4}{n^4}\cdot\left[\frac{n(n+1)}{2}\right]^2 \\ &= \lim_{n\to\infty}\frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \lim_{n\to\infty} 4\cdot\frac{(n+1)^2}{n^2} \\ &= 4\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = 4\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 4. \end{aligned}$$

(b) Verify your result using the Fundamental Theorem of Calculus.
Solution. Using the second part of the FTC, we obtain:

$$\int_0^2 x^3 dx = \frac{1}{4}x^4\Big|_0^2 = \frac{1}{4}(2)^4 - \frac{1}{4}(0)^4 = 4.$$

12. Find the **points** on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$. (Note that there may be more than one!)
Solution.

For any point (x, y) on the ellipse, the distance from $(1, 0)$ is given by the function $D = \sqrt{(x-1)^2 + (y-0)^2}$. Further, if (x, y) is on the ellipse, we have $y^2 = 4 - 4x^2$, and thus $D = \sqrt{(x-1)^2 + (4-4x^2)}$. To maximize D, we find the critical points. Taking the derivative of D with respect to x, we have $D' = \frac{1}{2}\frac{1}{\sqrt{(x-1)^2+4-4x^2}}(2(x-1) - 8x)$. This does not exist when $D = \sqrt{(x-1)^2 + (4-4x^2)} = 0$ which is a minimum for D, not a max. D' is zero when

$$2(x-1) - 8x = 0 \Rightarrow -6x - 2 = 0 \Rightarrow -6x = 2 \Rightarrow x = -1/3.$$

Plugging in $D'(0) = (\frac{1}{2})(\frac{1}{\sqrt{5}})(-2) < 0$, and $D'(-1) = (\frac{1}{2})(\frac{1}{4})(4) > 0$. So by the First Derivative Test, $x = -1/3$ is a local max, and since D is continuous and $x = -1/3$ is the only critical point on $[1, -1)$ it must be a global max.

We solve for y using $y^2 = 4 - 4x^2 = 4 - \frac{4}{9} = \frac{32}{9}$ and obtain $y = \frac{4}{3}\sqrt{2}$ or $y = -\frac{4}{3}\sqrt{2}$.

Hence, the farthest points are $(-\frac{1}{3}, \frac{4}{3}\sqrt{2})$ and $(-\frac{1}{3}, -\frac{4}{3}\sqrt{2})$.

13. Find the indefinite integral

$$\int (x^3(x^2+1)^2 + \cos x \sin x)dx.$$

Solution. We split up the integral to have $\int x^3(x^2+1)^2 dx + \int \cos x \, \sin x \, dx$.

Thus for the first half we get

$$\int x^3(x^2+1)^2 dx = \int (x^7 + 2x^5 + x^3) \, dx = \frac{x^8}{8} + \frac{2x^6}{6} + \frac{x^4}{4} + C_1. \tag{1}$$

For the second half, we can use u substitution. Letting $u = \sin x$ and thus $du = \cos x \, dx$, we get

$$\int \cos x \, \sin x \, dx = \int u du = \frac{u^2}{2} + C_2 = \frac{\sin^2 x}{2} + C_2. \tag{2}$$

Thus, we get

$$\frac{x^8}{8} + \frac{2x^6}{6} + \frac{x^4}{4} + \frac{\sin^2 x}{2} + C$$

as a final answer (where $C = C_1 + C_2$).

An alternate way of doing the first half is using substitution with $u = x^2 + 1$ and thus $du = 2xdx$. Then $x^2 = u - 1$, and

$$\begin{aligned}\int x^3(x^2+1)^2dx &= \int x^2(x^2+1)^2xdx = \int (u-1)u^2\frac{1}{2}du \\ &= \frac{1}{2}\int (u^3-u^2)du = \frac{1}{2}\left(\frac{u^4}{4}-\frac{u^3}{3}\right)+C_1 \\ &= \frac{1}{2}\left(\frac{(x^2+1)^4}{4}-\frac{(x^2+1)^3}{3}\right)+C_1. \qquad (3)\end{aligned}$$

Notice that Equation (1) and (3) just differ by a constant that is absorbed into the C_1.

An alternate way of doing the second half is using substitution with $u = \cos x$ and thus $du = -\sin x\ dx$ to get

$$\int \cos x\ \sin x\ dx = \frac{-\cos^2 x}{2} + C_2. \qquad (4)$$

However, these answers (2) and (4) are the same, since

$$\sin^2 x + \cos^2 x = 1$$

and so

$$\frac{\sin^2 x}{2} = -\frac{\cos^2 x}{2} + \frac{1}{2}.$$

Since we already have $+C_2$ at the end of the solution, the $\frac{1}{2}$ is "absorbed" into the C_2. Thus, both of our answers are correct.